



TITLE:

# On the Chern character of $SO(n)$ (Recent development of algebraic topology)

AUTHOR(S):

Watanabe, Takashi

---

CITATION:

Watanabe, Takashi. On the Chern character of  $SO(n)$ (Recent development of algebraic topology). 数理解析研究所講究録 1992, 781: 88-107

ISSUE DATE:

1992-04

URL:

<http://hdl.handle.net/2433/82503>

RIGHT:

On the Chern character of  $SO(n)$

大阪女子大 渡辺 孝 (Takashi Watanabe)

In [6] and [7] we described the Chern character homomorphism  $ch: K^*(G) \rightarrow H^{**}(G; \mathbb{Q})$  for  $G = Spin(2n+1)$  and  $SO(2n+1)$ . The purpose of this paper is to study  $ch$  for  $G = SO(2n)$  and  $Spin(2n)$ , where  $n \geq 1$ .

§1. Representation rings.

In this section, for later use, we quote from [3] and [8] some results on the complex representation rings of classical Lie groups that concern us.

Let  $G$  be a compact, connected Lie group. Its representation ring  $R(G)$  is the Grothendieck construction of the semiring of isomorphism classes  $[V]$  of  $G$ -modules  $V$  over  $\mathbb{C}$ . It has an augmentation

$$\varepsilon: R(G) \rightarrow \mathbb{Z}$$

which assigns to each class  $[V]$  the dimension of  $V$ . Let  $T$  be a maximal torus of  $G$ . The Weyl group  $W(G) = N(T)/T$  of  $G$  acts on  $T$  and therefore on  $R(T)$ . The inclusion  $i: T \rightarrow G$  induces a monomorphism  $i^*: R(G) \rightarrow R(T)$  and its image  $i^*(R(G))$  coincides with the subring  $R(T)^{W(G)}$  of elements left element-wise fixed by  $W(G)$ . Thus, through  $i^*$ ,  $R(G)$  can be regarded as a subring of  $R(T)$ . Besides,  $R(G)$  is a  $\lambda$ -ring with operations

$$\lambda^k: R(G) \rightarrow R(G) \quad \text{for } k \geq 0$$

induced by the exterior powers of  $G$ -modules over  $\mathbb{C}$ . Their properties needed in the sequel are:  $\lambda^0(x) = 1$  for all  $x \in R(G)$ ; if  $\varepsilon(x) = n$ , then  $\varepsilon(\lambda^k(x)) = \binom{n}{k}$  and  $\lambda^k(x) = 0$  for  $k > n$ .

Let  $T$  be the maximal torus of diagonal matrices in the unitary group  $U(n)$ . If  $\alpha_1, \dots, \alpha_n$  denote the standard 1-dimensional representations of  $T$ , then

$$(1.1) \quad R(T) = \mathbb{Z}[\alpha_1, \alpha_1^{-1}, \dots, \alpha_n, \alpha_n^{-1}] / (\alpha_1 \alpha_1^{-1} - 1, \dots, \alpha_n \alpha_n^{-1} - 1).$$

Put  $\lambda_1 = [C^n] \in R(U(n))$  and let  $\lambda_k = \lambda^k(\lambda_1)$ . Then

$$(1.2) \quad R(U(n)) = \mathbb{Z}[\lambda_1, \dots, \lambda_{n-1}, \lambda_n, \lambda_n^{-1}] / (\lambda_n \lambda_n^{-1} - 1)$$

and, as a subring of  $R(T)$ , the relation

$$(1.3) \quad \prod_{i=1}^n (1 + \alpha_i t) = \sum_{k=0}^n \lambda_k t^k$$

holds (in the polynomial ring  $R(T)[t]$ ).

For the rotation groups  $SO(n)$ , there is a fibre bundle

$$(1.4) \quad SO(n) \xrightarrow{j_n} SO(n+1) \xrightarrow{q_n} SO(n+1)/SO(n) = S^n.$$

Let

$$(1.5) \quad j'_n: U(n) \rightarrow SO(2n)$$

be the real restriction mapping. Then  $j'_n(T)$  becomes a maximal torus of  $SO(2n)$ , which we denote by  $T$  again. Further,  $j_{2n}(T)$  becomes a maximal torus of  $SO(2n+1)$ , we denote by  $T$  also. Put  $\mu'_1 = [(R^{2n+1})^C] \in R(SO(2n+1))$  and let  $\mu'_k = \lambda^k(\mu'_1)$ . Then

$$(1.6) \quad R(SO(2n+1)) = \mathbb{Z}[\mu'_1, \mu'_2, \dots, \mu'_n]$$

and, as a subring of  $R(T)$  (which is just as in (1.1)), the relation

$$(1.7) \quad (1+t) \prod_{i=1}^n (1 + \alpha_i t)(1 + \alpha_i^{-1} t) = \sum_{k=0}^{2n+1} \mu'_k t^k$$

holds. Note that  $\mu'_{2n+1-k} = \mu'_k$  for  $k = 0, 1, \dots, 2n+1$ ; in fact,

$$\begin{aligned} \sum_{k=0}^{2n+1} \mu'_{2n+1-k} t^k &= (t+1) \prod_{i=1}^n (t + \alpha_i)(t + \alpha_i^{-1}) \\ &= (t+1) \prod_{i=1}^n (\alpha_i^{-1}t + 1) \alpha_i \alpha_i^{-1} (\alpha_i t + 1) \\ &= (1+t) \prod_{i=1}^n (1 + \alpha_i^{-1}t)(1 + \alpha_i t) = \sum_{k=0}^{2n+1} \mu'_k t^k. \end{aligned}$$

For  $n \geq 3$ , the spinor group  $\text{Spin}(n)$  is a universal covering group of  $\text{SO}(n)$ :

$$(1.8) \quad S^0 = \{1, -1\} \longrightarrow \text{Spin}(n) \xrightarrow{p_n} \text{SO}(n).$$

Then  $p_n^{-1}(T)$  is a maximal torus of  $\text{Spin}(n)$ , which we denote by  $\tilde{T}$ . For this torus  $\tilde{T}$  of  $\text{Spin}(2n+1)$ , there are 1-dimensional representations  $\alpha_1, \dots, \alpha_n$  of  $\tilde{T}$  such that

$$(1.9) \quad R(\tilde{T}) = \mathbb{Z}[\alpha_1, \alpha_1^{-1}, \dots, \alpha_n, \alpha_n^{-1}, (\alpha_1 \cdots \alpha_n)^{1/2}] / \\ (\alpha_1 \alpha_1^{-1} - 1, \dots, \alpha_n \alpha_n^{-1} - 1, ((\alpha_1 \cdots \alpha_n)^{1/2})^2 - \alpha_1 \cdots \alpha_n)$$

and  $p_{2n+1}: \tilde{T} \rightarrow T$  induces the obvious inclusion  $R(T) \rightarrow R(\tilde{T})$  under the descriptions of (1.1) and (1.9). Set  $\mu'_k = p_{2n+1}^*(\mu_k) \in R(\text{Spin}(2n+1))$ . Let  $\Delta_{2n+1}$  be the spin representation of dimension  $2^n$ . Then

$$(1.10) \quad R(\text{Spin}(2n+1)) = \mathbb{Z}[\mu'_1, \mu'_2, \dots, \mu'_{n-1}, \Delta_{2n+1}]$$

and in  $R(\tilde{T})$ ,

$$(1.11) \quad \Delta_{2n+1} = \prod_{i=1}^n (\alpha_i^{1/2} + \alpha_i^{-1/2}) = \sum_{\varepsilon_i = \pm 1} \alpha_1^{\varepsilon_1/2} \cdots \alpha_n^{\varepsilon_n/2}$$

Moreover in  $R(\text{Spin}(2n+1))$ , the following relation holds:

$$(1.12) \quad \Delta_{2n+1}^2 = \sum_{k=0}^n \mu'_k.$$

Therefore,  $p_{2n+1}^*: R(\text{SO}(2n+1)) \rightarrow R(\text{Spin}(2n+1))$  is given by

$$(1.13) \quad p_{2n+1}^*(\mu'_k) = \mu_k \quad (k = 1, \dots, n-1),$$

$$p_{2n+1}^*(\mu'_n) = \Delta_{2n+1}^2 - \sum_{k=0}^{n-1} \mu'_k.$$

Put  $\mu_1 = [(R^{2n})^C] \in R(SO(2n))$  and let  $\mu_k = \lambda^k(\mu_1)$ . In particular,  $\mu_n$  can be halved, that is, there are two representations  $\mu_n^+, \mu_n^-$  of  $SO(2n)$  such that

$$(1.14) \quad \mu_n = \mu_n^+ + \mu_n^- \quad \text{and} \quad \varepsilon(\mu_n^+) = \varepsilon(\mu_n^-) = \frac{1}{2} \binom{2n}{n}.$$

Then

$$(1.15) \quad R(SO(2n)) = Z[\mu_1, \mu_2, \dots, \mu_{n-1}, \mu_n^+, \mu_n^-] / (\gamma_n)$$

where

$$\gamma_n = (\mu_n^+ + \sum_{i \geq 1} \mu_{n-2i})(\mu_n^- + \sum_{i \geq 1} \mu_{n-2i}) - (\mu_{n-1} + \sum_{j \geq 1} \mu_{n-1-2j}).$$

Here the summations in the right side end at  $\dots + \mu_4 + \mu_2 + 1$  or  $\dots + \mu_3 + \mu_1$ . As a subring of  $R(T)$  (which is just as in (1.1)), the relations

$$(1.16) \quad \prod_{i=1}^n (1 + \alpha_i t)(1 + \alpha_i^{-1} t) = \sum_{k=0}^{2n} \mu_k t^k$$

and

$$(1.17) \quad \begin{aligned} \mu_n^+ &= \sum_{\prod \varepsilon_i = 1} \alpha_1^{\varepsilon_1} \dots \alpha_n^{\varepsilon_n} + \sum'' \alpha_1^{\varepsilon_1} \dots \hat{\alpha}_i^{\varepsilon_i} \dots \hat{\alpha}_j^{\varepsilon_j} \dots \alpha_n^{\varepsilon_n}, \\ \mu_n^- &= \sum_{\prod \varepsilon_i = -1} \alpha_1^{\varepsilon_1} \dots \alpha_n^{\varepsilon_n} + \sum'' \alpha_1^{\varepsilon_1} \dots \hat{\alpha}_i^{\varepsilon_i} \dots \hat{\alpha}_j^{\varepsilon_j} \dots \alpha_n^{\varepsilon_n} \end{aligned}$$

hold, where the notation  $\hat{\alpha}^{\varepsilon}$  means the replacement of  $\alpha^{\varepsilon}$  by 1 and the number of  $\hat{\alpha}$  in the summation  $\sum''$  is even and positive.

Set  $\mu_k = p_{2n}^*(\mu_k) \in R(\text{Spin}(2n))$ . Let  $\Delta_{2n}^+, \Delta_{2n}^-$  be the half spin representations, each of dimension  $2^{n-1}$ . Then

$$(1.18) \quad R(\text{Spin}(2n)) = Z[\mu_1, \mu_2, \dots, \mu_{n-2}, \Delta_{2n}^+, \Delta_{2n}^-]$$

and in  $R(\tilde{T})$  (which is just as in (1.9)),

$$(1.19) \quad \Delta_{2n}^+ = \sum_{\prod \varepsilon_i = 1} \alpha_1^{\varepsilon_1/2} \dots \alpha_n^{\varepsilon_n/2}, \quad \Delta_{2n}^- = \sum_{\prod \varepsilon_i = -1} \alpha_1^{\varepsilon_1/2} \dots \alpha_n^{\varepsilon_n/2}.$$

Moreover in  $R(\text{Spin}(2n))$ , the following relations hold:

$$(1.20) \quad \begin{aligned} \Delta_{2n}^+ \Delta_{2n}^- &= \mu_{n-1} + \sum_{q \geq 1} \mu_{n-1-2q}, \\ \Delta_{2n}^{+2} &= \mu_n^+ + \sum_{k \geq 1} \mu_{n-2k}, \\ \Delta_{2n}^{-2} &= \mu_n^- + \sum_{k \geq 1} \mu_{n-2k}. \end{aligned}$$

Therefore,  $p_{2n}^*: R(\text{SO}(2n)) \rightarrow R(\text{Spin}(2n))$  is given by

$$(1.21) \quad \begin{aligned} p_{2n}^*(\mu_k) &= \mu_k \quad (k = 1, \dots, n-2), \\ p_{2n}^*(\mu_{n-1}) &= \Delta_{2n}^+ \Delta_{2n}^- - \sum_{q \geq 1} \mu_{n-1-2q}, \\ p_{2n}^*(\mu_n^+) &= \Delta_{2n}^{+2} - \sum_{k \geq 1} \mu_{n-2k}, \\ p_{2n}^*(\mu_n^-) &= \Delta_{2n}^{-2} - \sum_{k \geq 1} \mu_{n-2k}. \end{aligned}$$

The following propositions are immediate consequences of the above results.

**Proposition 1.1.** (1)  $j_{2n}^*: R(\text{SO}(2n+1)) \rightarrow R(\text{SO}(2n))$  satisfies

$$j_{2n}^*(\mu_1') = \mu_1 + 1.$$

(2)  $j_{2n-1}^*: R(\text{SO}(2n)) \rightarrow R(\text{SO}(2n-1))$  satisfies

$$j_{2n-1}^*(\mu_1) = \mu_1' + 1.$$

**Proposition 1.2.** (1)  $\tilde{j}_{2n}^*: R(\text{Spin}(2n+1)) \rightarrow R(\text{Spin}(2n))$  satisfies

$$\begin{aligned} \tilde{j}_{2n}^*(\mu_1') &= \mu_1 + 1, \\ \tilde{j}_{2n}^*(\Delta_{2n+1}^+) &= \Delta_{2n}^+ + \Delta_{2n}^-. \end{aligned}$$

(2)  $\tilde{j}_{2n-1}^*: R(\text{Spin}(2n)) \rightarrow R(\text{Spin}(2n-1))$  satisfies

$$\begin{aligned} \tilde{j}_{2n-1}^*(\mu_1) &= \mu_1' + 1, \\ \tilde{j}_{2n-1}^*(\Delta_{2n}^+) &= \tilde{j}_{2n-1}^*(\Delta_{2n}^-) = \Delta_{2n-1}. \end{aligned}$$

**Proposition 1.3.**  $j_n'^*: R(SO(2n)) \rightarrow R(U(n))$  satisfies

$$j_n'^*(u_k) = \lambda_n^{-1} \sum_{i=0}^k \lambda_i \lambda_{n-k+i} \quad (k = 1, \dots, n).$$

**Proposition 1.4.** In  $R(T)$ ,

$$\Delta_{2n}^{+2} - \Delta_{2n}^{-2} = \lambda_n^{-1} \left( \sum_{i=0}^n \lambda_i \right) \left( \sum_{j=0}^n (-1)^{n-j} \lambda_j \right).$$

## §2. Cohomology rings

In this section we fix some notations concerning the integral cohomology of our groups  $G$ .

For  $G = U(n)$ , by Borel's transgression theorem, there exist elements  $x_{2i-1} \in H^{2i-1}(U(n); \mathbb{Z})$ ,  $i = 1, \dots, n$ , such that

$$H^*(U(n); \mathbb{Z}) = \Lambda_{\mathbb{Z}}(x_1, x_3, \dots, x_{2n-1})$$

and

$$PH^*(U(n); \mathbb{Z}) = \mathbb{Z}\{x_1, x_3, \dots, x_{2n-1}\}$$

where  $P$  denotes the primitive module functor. Thus

$$(2.1) \quad H^*(U(n); \mathbb{Q}) = \Lambda_{\mathbb{Q}}(x_1, x_3, \dots, x_{2n-1}).$$

$H^*(SO(n); \mathbb{Z})$  has 2-torsion only. For  $G = SO(2n+1)$ , by Poincaré duality and Borel's transgression theorem, there exist elements  $x_{4i-1} \in H^{4i-1}(SO(2n+1); \mathbb{Z})$ ,  $i = 1, \dots, n$ , such that

$$H^*(SO(2n+1); \mathbb{Z})/\text{Tor} = \Lambda_{\mathbb{Z}}(x_3, x_7, \dots, x_{4n-1})$$

and

$$PH^*(SO(2n+1); \mathbb{Q}) = \mathbb{Q}\{x_3, x_7, \dots, x_{4n-1}\}.$$

Thus

$$(2.2) \quad H^*(SO(2n+1); \mathbb{Q}) = \Lambda_{\mathbb{Q}}(x_3, x_7, \dots, x_{4n-1}).$$

$H^*(\text{Spin}(n); \mathbb{Z})$  has 2-torsion if and only if  $n \geq 7$ . For  $G = \text{Spin}(2n+1)$ , by similar reasons, there exist elements  $\tilde{x}_{4i-1} \in H^{4i-1}(\text{Spin}(2n+1); \mathbb{Z})$ ,  $i = 1, \dots, n$ , such that

$$H^*(\text{Spin}(2n+1); \mathbb{Z})/\text{Tor} = \Lambda_{\mathbb{Z}}(\tilde{x}_3, \tilde{x}_7, \dots, \tilde{x}_{4n-1})$$

and

$$PH^*(\text{Spin}(2n+1); \mathbb{Q}) = \mathbb{Q}\{\tilde{x}_3, \tilde{x}_7, \dots, \tilde{x}_{4n-1}\}.$$

Thus

$$(2.3) \quad H^*(\text{Spin}(2n+1); \mathbb{Q}) = \Lambda_{\mathbb{Q}}(\tilde{x}_3, \tilde{x}_7, \dots, \tilde{x}_{4n-1}).$$

For  $G = \text{SO}(2n)$ , there exist elements  $x_{4i-1} \in H^{4i-1}(\text{SO}(2n); \mathbb{Z})$ ,  $i = 1, \dots, n-1$ , and  $x'_{2n-1} \in H^{2n-1}(\text{SO}(2n); \mathbb{Z})$  such that

$$H^*(\text{SO}(2n); \mathbb{Z})/\text{Tor} = \Lambda_{\mathbb{Z}}(x_3, x_7, \dots, x_{4n-5}, x'_{2n-1})$$

and

$$(2.4) \quad PH^*(\text{SO}(2n); \mathbb{Q}) = \mathbb{Q}\{x_3, x_7, \dots, x_{4n-5}, x'_{2n-1}\}.$$

Thus

$$(2.5) \quad H^*(\text{SO}(2n); \mathbb{Q}) = \Lambda_{\mathbb{Q}}(x_3, x_7, \dots, x_{4n-5}, x'_{2n-1}).$$

For  $G = \text{Spin}(2n)$ , there exist elements  $\tilde{x}_{4i-1} \in H^{4i-1}(\text{Spin}(2n); \mathbb{Z})$ ,  $i = 1, \dots, n-1$ , and  $\tilde{x}'_{2n-1} \in H^{2n-1}(\text{Spin}(2n); \mathbb{Z})$  such that

$$H^*(\text{Spin}(2n); \mathbb{Z})/\text{Tor} = \Lambda_{\mathbb{Z}}(\tilde{x}_3, \tilde{x}_7, \dots, \tilde{x}_{4n-5}, \tilde{x}'_{2n-1})$$

and

$$PH^*(\text{Spin}(2n); \mathbb{Q}) = \mathbb{Q}\{\tilde{x}_3, \tilde{x}_7, \dots, \tilde{x}_{4n-5}, \tilde{x}'_{2n-1}\}.$$

Thus

$$(2.6) \quad H^*(\text{Spin}(2n); \mathbb{Q}) = \Lambda_{\mathbb{Q}}(\tilde{x}_3, \tilde{x}_7, \dots, \tilde{x}_{4n-5}, \tilde{x}'_{2n-1}).$$

The following two propositions are easy.

**Proposition 2.1.**  $j_{2n-1}^*: H^*(\text{SO}(2n); \mathbb{Z}) \rightarrow H^*(\text{SO}(2n-1); \mathbb{Z})$  satisfies

$$j_{2n-1}^*(x_{4i-1}) = x_{4i-1} \quad (i = 1, \dots, n-1),$$



$$j_{2n-1}^*(x'_{2n-1}) = 0.$$

**Proposition 2.2.**  $\tilde{j}_{2n-1}^*: H^*(\text{Spin}(2n); \mathbb{Z}) \rightarrow H^*(\text{Spin}(2n-1); \mathbb{Z})$  satisfies

$$\begin{aligned}\tilde{j}_{2n-1}^*(\tilde{x}_{4i-1}) &= \tilde{x}_{4i-1} \quad (i = 1, \dots, n-1), \\ \tilde{j}_{2n-1}^*(\tilde{x}'_{2n-1}) &= 0.\end{aligned}$$

**Proposition 2.3.**  $j_n^*: H^*(\text{SO}(2n); \mathbb{Z}) \rightarrow H^*(\text{U}(n); \mathbb{Z})$  satisfies

$$j_n^*(x'_{2n-1}) = x_{2n-1}.$$

Furthermore, when  $n = 2m$ , not only  $j_{2m}^*(x'_{4m-1}) = x_{4m-1}$ , but also  $j_{2m}^*(x_{4m-1}) = 0$ .

**Proof.** Since

$$\begin{aligned}H^*(\text{SO}(2n)/\text{U}(n); \mathbb{Z}) &= \mathbb{Z}[e_2, e_4, \dots, e_{2n-2}]/ \\ &\quad (e_{4k} + \sum_{i=1}^{2k-1} (-1)^i e_{2i} e_{4k-2i} : k = 1, \dots, n-1) \\ &= \Delta_{\mathbb{Z}}(e_2, e_4, \dots, e_{2n-2})\end{aligned}$$

where  $\Delta_R(\ )$  denotes the algebra over a ring  $R$  having a simple system of generators, and  $e_{2i} \in H^{2i}(\text{SO}(2n)/\text{U}(n); \mathbb{Z})$  (see [4, Chapter 3, Theorem 6.11]), the result follows from the spectral sequence argument for the integral cohomology of the fibre bundle

$$\text{U}(n) \xrightarrow{j_n'} \text{SO}(2n) \xrightarrow{q_n'} \text{SO}(2n)/\text{U}(n).$$

### §3. K-rings

In this section we collect some results on the complex K-theory of our groups  $G$ .

Let

$$\mathcal{B}: R(G) \rightarrow K^{-1}(G)$$

be the map of [2]; then  $\mathcal{B}$  has the following properties:

(3.1)(1) For each  $\rho_1, \rho_2 \in R(G)$ ,

$$\mathcal{B}(\rho_1 + \rho_2) = \mathcal{B}(\rho_1) + \mathcal{B}(\rho_2);$$

(2) If  $n \in R(G)$  is the class of a trivial  $G$ -module of dimension  $n$ , then  $\mathcal{B}(n) = 0$ ;

(3) For each  $\rho_1, \rho_2 \in R(G)$ ,

$$\mathcal{B}(\rho_1 \rho_2) = \varepsilon(\rho_2) \mathcal{B}(\rho_1) + \varepsilon(\rho_1) \mathcal{B}(\rho_2).$$

As will be seen below, the  $\mathbb{Z}/(2)$ -graded  $K$ -rings  $K^*(G)$  can be described by using this map  $\mathcal{B}$ . Indeed, Hodgkin's theorem [2, Theorem A] says that if  $G$  is a compact connected Lie group with  $\pi_1(G)$  torsion-free, then  $K^*(G)$  is torsion-free and has the structure of a Hopf algebra over  $\mathbb{Z}$ ; more precisely, if

$$R(G) = \mathbb{Z}[\rho_1, \rho_2, \dots, \rho_q],$$

then

$$K^*(G) = \Lambda_{\mathbb{Z}}(\mathcal{B}(\rho_1), \mathcal{B}(\rho_2), \dots, \mathcal{B}(\rho_q))$$

where each  $\mathcal{B}(\rho_i)$  is primitive. Therefore, for  $G = U(n)$ , it follows from (1.2) and (3.1) that  $\mathcal{B}(\lambda_n^{-1}) = -\mathcal{B}(\lambda_n)$  and

$$(3.2) \quad K^*(U(n)) = \Lambda_{\mathbb{Z}}(\mathcal{B}(\lambda_1), \dots, \mathcal{B}(\lambda_{n-1}), \mathcal{B}(\lambda_n)).$$

Similarly, for  $G = \text{Spin}(2n+1)$ , it follows from (1.10) that

$$(3.3) \quad K^*(\text{Spin}(2n+1)) = \Lambda_{\mathbb{Z}}(\mathcal{B}(\mu'_1), \dots, \mathcal{B}(\mu'_{n-1}), \mathcal{B}(\Delta_{2n+1})).$$

For  $G = \text{SO}(2n+1)$ , by (1.6),  $\mathcal{B}(\mu'_1), \dots, \mathcal{B}(\mu'_n) \in K^{-1}(\text{SO}(2n+1))$ . According to [1], there exist other two elements  $\varepsilon_{2n+1} \in K^{-1}(\text{SO}(2n+1))$  and  $\varepsilon_{2n+1} \in K^0(\text{SO}(2n+1))$  such that

$$(3.4) \quad p_{2n+1}^*(\varepsilon_{2n+1}) = 2\mathcal{B}(\Delta_{2n+1})$$

and

$$K^*(\text{SO}(2n+1)) = \Lambda_{\mathbb{Z}}(\mathcal{B}(\mu'_1), \dots, \mathcal{B}(\mu'_{n-1}), \varepsilon_{2n+1}) \otimes$$

$$(\mathbb{Z}\{1\} \oplus \mathbb{Z}/(2^n)\{\varepsilon_{2n+1}\})/(\varepsilon_{2n+1} \otimes \varepsilon_{2n+1}).$$

Thus

$$(3.5) \quad K^*(SO(2n+1))/\text{Tor} = \Lambda_{\mathbb{Z}}(\beta(u'_1), \dots, \beta(u'_{n-1}), \varepsilon_{2n+1}).$$

Similarly, for  $G = \text{Spin}(2n)$ , it follows from (1.18) that

$$(3.6) \quad K^*(\text{Spin}(2n)) = \Lambda_{\mathbb{Z}}(\beta(u_1), \dots, \beta(u_{n-2}), \beta(\Delta_{2n}^+), \beta(\Delta_{2n}^-)).$$

For  $G = SO(2n)$ , by (1.15),  $\beta(u_1), \dots, \beta(u_{n-1}), \beta(u_n^+), \beta(u_n^-) \in K^{-1}(SO(2n))$ . According to [1], there exist other three elements  $\delta_{2n}, \varepsilon_{2n} \in K^{-1}(SO(2n))$  and  $\varepsilon_{2n} \in K^0(SO(2n))$  such that

$$(3.7) \quad p_{2n}^*(\delta_{2n}) = \beta(\Delta_{2n}^+) - \beta(\Delta_{2n}^-), \quad p_{2n}^*(\varepsilon_{2n}) = 2\beta(\Delta_{2n}^+)$$

and

$$K^*(SO(2n)) = \Lambda_{\mathbb{Z}}(\beta(u_1), \dots, \beta(u_{n-2}), \delta_{2n}, \varepsilon_{2n}) \otimes (\mathbb{Z}\{1\} \oplus \mathbb{Z}/(2^{n-1})\{\varepsilon_{2n}\})/(\varepsilon_{2n} \otimes \varepsilon_{2n}).$$

Thus

$$(3.8) \quad K^*(SO(2n))/\text{Tor} = \Lambda_{\mathbb{Z}}(\beta(u_1), \dots, \beta(u_{n-2}), \delta_{2n}, \varepsilon_{2n}).$$

Using these results, we can deduce the following from the results of §1.

**Proposition 3.1.** (1) In  $K^*(\text{Spin}(2n+1))$ ,

$$\beta(u'_n) = 2^{n+1}\beta(\Delta_{2n+1}) - \sum_{k=1}^{n-1} \beta(u'_k).$$

(2) In  $K^*(\text{Spin}(2n))$ ,

$$\beta(u_{n-1}) = 2^{n-1}\beta(\Delta_{2n}^+) + 2^{n-1}\beta(\Delta_{2n}^-) - \sum_{q=1}^{[(n-2)/2]} \beta(u_{n-1-2q}).$$

**Proposition 3.2.** (1) In  $K^*(SO(2n+1))/\text{Tor}$ ,

$$\beta(u'_n) = 2^n \varepsilon_{2n+1} - \sum_{k=1}^{n-1} \beta(u'_k).$$

(2) In  $K^*(SO(2n))/\text{Tor}$ ,

$$\begin{aligned}
\mathcal{B}(\mu_{n-1}) &= -2^{n-1}\delta_{2n} + 2^{n-1}\varepsilon_{2n} - \sum_{q=1}^{[(n-2)/2]} \mathcal{B}(\mu_{n-1-2q}), \\
\mathcal{B}(\mu_n^+) &= 2^{n-1}\varepsilon_{2n} - \sum_{k=1}^{[(n-1)/2]} \mathcal{B}(\mu_{n-2k}), \\
\mathcal{B}(\mu_n^-) &= -2^n\delta_{2n} + 2^{n-1}\varepsilon_{2n} - \sum_{k=1}^{[(n-1)/2]} \mathcal{B}(\mu_{n-2k}).
\end{aligned}$$

**Proposition 3.3.** (1)  $\tilde{j}_{2n}^*: K^*(\text{Spin}(2n+1)) \rightarrow K^*(\text{Spin}(2n))$  satisfies

$$\begin{aligned}
\tilde{j}_{2n}^*(\mathcal{B}(\mu_1')) &= \mathcal{B}(\mu_1), \\
\tilde{j}_{2n}^*(\mathcal{B}(\Delta_{2n+1}^+)) &= \mathcal{B}(\Delta_{2n}^+) + \mathcal{B}(\Delta_{2n}^-), \\
(2) \quad \tilde{j}_{2n-1}^*: K^*(\text{Spin}(2n)) &\rightarrow K^*(\text{Spin}(2n-1)) \text{ satisfies} \\
\tilde{j}_{2n-1}^*(\mathcal{B}(\mu_1')) &= \mathcal{B}(\mu_1'), \\
\tilde{j}_{2n-1}^*(\mathcal{B}(\Delta_{2n}^+)) &= \tilde{j}_{2n-1}^*(\mathcal{B}(\Delta_{2n}^-)) = \mathcal{B}(\Delta_{2n-1}).
\end{aligned}$$

**Proposition 3.4.** (1)  $j_{2n}^*: K^*(\text{SO}(2n+1))/\text{Tor} \rightarrow K^*(\text{SO}(2n))/\text{Tor}$  satisfies

$$\begin{aligned}
j_{2n}^*(\mathcal{B}(\mu_1')) &= \mathcal{B}(\mu_1), \\
j_{2n}^*(\varepsilon_{2n+1}) &= -2\delta_{2n} + 2\varepsilon_{2n}, \\
(2) \quad j_{2n-1}^*: K^*(\text{SO}(2n))/\text{Tor} &\rightarrow K^*(\text{SO}(2n-1))/\text{Tor} \text{ satisfies} \\
j_{2n-1}^*(\mathcal{B}(\mu_1')) &= \mathcal{B}(\mu_1'), \\
j_{2n-1}^*(\delta_{2n}) &= 0, \quad j_{2n-1}^*(\varepsilon_{2n}) = \varepsilon_{2n-1}.
\end{aligned}$$

**Proposition 3.5.**  $j_n'^*: K^*(\text{SO}(2n))/\text{Tor} \rightarrow K^*(U(n))$  satisfies

$$\begin{aligned}
j_n'^*(\mathcal{B}(\mu_1)) &= \mathcal{B}(\lambda_1) + \mathcal{B}(\lambda_{n-1}) - n\mathcal{B}(\lambda_n), \\
j_n'^*(\delta_{2n}) &= \sum_{k=1}^n (-1)^{n-k} \mathcal{B}(\lambda_k), \\
j_n'^*(\varepsilon_{2n}) &= \sum_{k=1}^{n-1} (1+(-1)^{n-k}) \mathcal{B}(\lambda_k) - 2(2^{n-2}-1)\mathcal{B}(\lambda_n).
\end{aligned}$$

#### §4. The Chern character homomorphisms

In this section we prove our main results.

Let  $\phi: \mathbb{N} \times \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{Z}$  be a function defined by

$$(4.1) \quad \phi(n, k, q) = \sum_{j=1}^k (-1)^{j-1} \binom{n}{k-j} j^{q-1} \quad \text{for } n, k, q \in \mathbb{N}.$$

Then, following Method I of [5, pp. 464-466] and using Lemma 1 of [5], we have

**Proposition 4.1.** In the notations of (3.2) and (2.1),  $\text{ch}$ :

$K^*(U(n)) \rightarrow H^{**}(U(n); \mathbb{Q})$  is given by

$$\text{ch}(\beta(\lambda_k)) = \sum_{i=1}^n \frac{(-1)^{i-1}}{(i-1)!} \phi(n, k, i) x_{2i-1} \quad (k \geq 1).$$

Let  $\mathcal{P} = \{2^i \mid i = 0, 1, 2, \dots\}$ . For each  $n \in \mathbb{N}$  there is a unique integer  $s(n)$  such that  $2^{s(n)-1} < n \leq 2^{s(n)}$ . Let

$$q(n, i) = \begin{cases} 2 & \text{if } n \notin \mathcal{P}, i = 2^{s(n)-1} \\ & \text{or } n \in \mathcal{P}, i = 2^{s(n)} \\ 1 & \text{otherwise} \end{cases}$$

Then Theorem 1 of [6] is

**Theorem 4.2.** In the notations of (3.3) and (2.3),  $\text{ch}: K^*(\text{Spin}(2n+1)) \rightarrow H^{**}(\text{Spin}(2n+1); \mathbb{Q})$  is given by

$$\begin{aligned} \text{ch}(\beta(\tilde{\mu}'_k)) &= \sum_{i=1}^n \frac{(-1)^{i-1} 2^{q(n, i)}}{(2i-1)!} \phi(2n+1, k, 2i) \tilde{x}_{4i-1}, \\ \text{ch}(\beta(\Delta_{2n+1})) &= \sum_{i=1}^n \frac{(-1)^{i-1} 2^{q(n, i)}}{(2i-1)!} \left( \frac{1}{2^{n+1}} \sum_{k=1}^n \phi(2n+1, k, 2i) \right) \tilde{x}_{4i-1}. \end{aligned}$$

While, Theorem 5.3 of [7] is

**Theorem 4.3.** In the notations of (3.5) and (2.2),  $\text{ch}: K^*(SO(2n+1))/\text{Tor} \rightarrow H^{**}(SO(2n+1); \mathbb{Q})$  is given by

$$\begin{aligned}\text{ch}(\beta(u'_k)) &= \sum_{i=1}^n \frac{(-1)^{i-1} 2}{(2i-1)!} \phi(2n+1, k, 2i) x_{4i-1}, \\ \text{ch}(\varepsilon_{2n+1}) &= \sum_{i=1}^n \frac{(-1)^{i-1} 2}{(2i-1)!} \left( \frac{1}{2^n} \sum_{k=1}^n \phi(2n+1, k, 2i) \right) x_{4i-1}.\end{aligned}$$

**Corollary 4.4.** (1)  $p_{2n+1}^*: H^*(SO(2n+1); \mathbb{Z}) \rightarrow H^*(\text{Spin}(2n+1); \mathbb{Z})$  satisfies

$$p_{2n+1}^*(x_{4i-1}) = 2^{q(n,i)-1} \tilde{x}_{4i-1} \quad (i = 1, \dots, n).$$

(2)  $p_{2n}^*: H^*(SO(2n); \mathbb{Z}) \rightarrow H^*(\text{Spin}(2n); \mathbb{Z})$  satisfies

$$\begin{aligned}p_{2n}^*(x_{4i-1}) &= 2^{q(n-1,i)-1} \tilde{x}_{4i-1} \quad (i = 1, \dots, n-1), \\ p_{2n}^*(x'_{2n-1}) &= \tilde{x}'_{2n-1}.\end{aligned}$$

**Theorem 4.5.** In the notations of (3.8) and (2.5),  $\text{ch}: K^*(SO(2n))/\text{Tor} \rightarrow H^{**}(SO(2n); \mathbb{Q})$  is given by

$$\begin{aligned}\text{ch}(\beta(u_k)) &= \sum_{i=1}^{n-1} \frac{(-1)^{i-1} 2}{(2i-1)!} \phi(2n, k, 2i) x_{4i-1} + \\ &\quad \frac{(-1)^{n-1} (1+(-1)^n)}{(n-1)!} \phi(2n, k, n) x'_{2n-1}, \\ \text{ch}(\delta_{2n}) &= x'_{2n-1}, \\ \text{ch}(\varepsilon_{2n}) &= \sum_{i=1}^{n-1} \frac{(-1)^{i-1} 2}{(2i-1)!} \left( \frac{1}{2^{n-1}} \sum_{q=0}^{[(n-2)/2]} \phi(2n, n-1-2q, 2i) \right) x_{4i-1} + \\ &\quad \left( 1 + \frac{(-1)^{n-1} (1+(-1)^n)}{(n-1)!} \right) \left( \frac{1}{2^{n-1}} \sum_{q=0}^{[(n-2)/2]} \phi(2n, n-1-2q, n) \right) x'_{2n-1}.\end{aligned}$$

**Proof.** Since  $\beta(u_1)$  is primitive in the Hopf algebra  $K^*(SO(2n))/\text{Tor}$  (see [1]), by (2.4) we may set

$$(4.2) \quad \text{ch}(\beta(u_1)) = \sum_{i=1}^{n-1} a_i x_{4i-1} + a' x'_{2n-1}$$

for some  $a_i, a' \in \mathbb{Q}$ . Let us compute these coefficients. Apply  $j_{2n-1}^*$  to (4.2). Then the left hand side is

$$\begin{aligned} j_{2n-1}^* \text{ch}(\mathcal{B}(u_1)) &= \text{ch}(j_{2n-1}^*(\mathcal{B}(u_1))) \\ &= \text{ch}(\mathcal{B}(u'_1)) && \text{by Proposition 3.4(2)} \\ &= \sum_{i=1}^{n-1} \frac{(-1)^{i-1} 2}{(2i-1)!} x_{4j-1} && \text{by Theorem 4.3,} \end{aligned}$$

and the right hand side is

$$j_{2n-1}^* \left( \sum_{i=1}^{n-1} a_i x_{4i-1} + a' x'_{2n-1} \right) = \sum_{i=1}^{n-1} a_i x_{4i-1}$$

by Proposition 2.1. Hence  $a_i = (-1)^{i-1} 2 / (2i-1)!$  for  $i = 1, \dots, n-1$ . Apply  $j_n^*$  to (4.2). Then the left hand side is

$$\begin{aligned} j_n^* \text{ch}(\mathcal{B}(u_1)) &= \text{ch}(j_n^*(\mathcal{B}(u_1))) \\ &= \text{ch}(\mathcal{B}(\lambda_1) + \mathcal{B}(\lambda_{n-1}) - n\mathcal{B}(\lambda_n)) && \text{by Proposition 3.5} \\ &= \sum_{i=1}^n \frac{(-1)^{i-1}}{(i-1)!} (1 + \phi(n, n-1, i) - n\phi(n, n, i)) x_{2i-1} \end{aligned}$$

by Proposition 4.1. Since  $\phi(n, k, 1) = \binom{n-1}{k-1}$ , we have

$$1 + \phi(n, n-1, 1) - n\phi(n, n, 1) = 1 + (n-1) - n \cdot 1 = 0$$

If  $2 \leq i \leq n$ , then  $\phi(n, n-k, i) = (-1)^i \phi(n, k, i)$  and  $\phi(n, k, i) = 0$  for  $k \geq n$  by (4.14) and (4.15) of [7]. Therefore

$$\begin{aligned} 1 + \phi(n, n-1, i) - n\phi(n, n, i) &= 1 + (-1)^i \phi(n, 1, i) - 0 \\ &= 1 + (-1)^i. \end{aligned}$$

Consequently, for  $i = 1, \dots, n$ ,

$$1 + \phi(n, n-1, i) - n\phi(n, n, i) = 1 + (-1)^i.$$

Thus

$$j_n^* \text{ch}(\mathcal{B}(u_1)) = \dots + \frac{(-1)^{n-1} (1 + (-1)^n)}{(n-1)!} x_{2n-1}.$$

On the other hand, the right hand side is

$$j_n^* \left( \sum_{i=1}^{n-1} a_i x_{4i-1} + a' x'_{2n-1} \right) = \dots + a' x_{2n-1}$$

by Proposition 2.3. Hence  $a' = (-1)^{n-1}(1+(-1)^n)/(n-1)!$ . This proves the first equality for  $k = 1$ , and that for  $k > 1$  follows from it and Lemma 1 of [5].

Next we set

$$(4.3) \quad \text{ch}(\delta_{2n}) = \sum_{i=1}^{n-1} a_i x_{4i-1} + a' x'_{2n-1}$$

for some  $a_i, a' \in \mathbb{Q}$ . Let us compute these coefficients. Apply  $j_{2n-1}^*$  to (4.3). Then the left hand side is

$$j_{2n-1}^* \text{ch}(\delta_{2n}) = \text{ch} j_{2n-1}^*(\delta_{2n}) = \text{ch}(0) = 0$$

by Proposition 3.4(2), and the right hand side is

$$j_{2n-1}^* \left( \sum_{i=1}^{n-1} a_i x_{4i-1} + a' x'_{2n-1} \right) = \sum_{i=1}^{n-1} a_i x_{4i-1}$$

by Proposition 2.1. Hence  $a_i = 0$  for  $i = 1, \dots, n-1$ . Apply

$j_n^*$  to (4.3). Then the left hand side is

$$\begin{aligned} j_n^* \text{ch}(\delta_{2n}) &= \text{ch}(j_n^*(\delta_{2n})) \\ &= \text{ch} \left( \sum_{k=1}^n (-1)^{n-k} \beta(\lambda_k) \right) && \text{by Proposition 3.5} \\ &= \sum_{k=1}^n (-1)^{n-k} \text{ch}(\beta(\lambda_k)) \\ &= \sum_{k=1}^n (-1)^{n-k} \left( \sum_{i=1}^n \frac{(-1)^{i-1}}{(i-1)!} \phi(n, k, i) x_{2i-1} \right) && \text{by Proposition 4.1} \\ &= \sum_{i=1}^n \frac{(-1)^{i-1}}{(i-1)!} \left( \sum_{k=1}^n (-1)^{n-k} \phi(n, k, i) \right) x_{2i-1}. \end{aligned}$$

But

$$\begin{aligned} \sum_{k=1}^n (-1)^{n-k} \phi(n, k, i) &= \sum_{k=1}^n (-1)^{n-k} \left( \sum_{j=1}^k (-1)^{j-1} \binom{n}{k-j} j^{i-1} \right) && \text{by (4.1)} \\ &= \sum_{j=1}^n \sum_{k=j}^n (-1)^{n-k+j-1} \binom{n}{k-j} j^{i-1} \\ &= \sum_{j=1}^n \sum_{\ell=0}^{n-j} (-1)^{n-\ell-1} \binom{n}{\ell} j^{i-1} \\ &= (-1)^{n-1} \sum_{j=1}^n \left( \sum_{\ell=0}^{n-j} (-1)^{\ell} \binom{n}{\ell} \right) j^{i-1} \end{aligned}$$



$$\begin{aligned}
&= (-1)^{n-1} \sum_{j=1}^n (-1)^{n-j} \binom{n-1}{n-j} j^{i-1} \\
&= \sum_{j=1}^n (-1)^{j-1} \binom{n-1}{n-j} j^{i-1} \\
&= \Phi(n-1, n, i) \\
&= \begin{cases} 0 & \text{if } i = 1, \dots, n-1 \\ (-1)^{n-1} (n-1)! & \text{if } i = n \end{cases} \quad \text{by [7].}
\end{aligned}$$

Thus

$$j_n'^* \text{ch}(\delta_{2n}) = \frac{(-1)^{n-1}}{(n-1)!} (-1)^{n-1} (n-1)! x_{2n-1} = x_{2n-1}.$$

On the other hand, the right hand side is

$$j_n'^* \left( \sum_{i=1}^{n-1} a_i x_{4i-1} + a' x_{2n-1} \right) = a' x_{2n-1}$$

by Proposition 2.3. Hence  $a' = 1$ . This proves the second equality.

The third equality is obtained from the first and second equalities by using the relation

$$2^{n-1} \varepsilon_{2n} = 2^{n-1} \delta_{2n} + \sum_{q=0}^{[(n-2)/2]} \beta(u_{n-1-2q})$$

of Proposition 3.2(2).

**Corollary 4.6.**  $j_n'^*: H^*(SO(2n); \mathbb{Z}) \rightarrow H^*(U(n); \mathbb{Z})$  satisfies

$$j_n'^*(x_{4i-1}) = \begin{cases} (-1)^i x_{4i-1} & \text{if } i = 1, \dots, [(n-1)/2] \\ 0 & \text{if } i = [(n-1)/2] + 1, \dots, n-1. \end{cases}$$

**Corollary 4.7.**  $j_{2n}^*: H^*(SO(2n+1); \mathbb{Z}) \rightarrow H^*(SO(2n); \mathbb{Z})$  satisfies:

(i) if  $n = 2m + 1$ ,

$$x_{4i-1} \quad (i = 1, \dots, 2m)$$

$$j_{4m+2}^*(x_{4i-1}) = \begin{cases} 0 & (i = 2m+1) \end{cases};$$

(ii) if  $n = 2m$ ,

$$j_{4m}^*(x_{4i-1}) = \begin{cases} x_{4i-1} & (i = 1, \dots, 2m-1 \text{ and } i \neq m) \\ x_{4m-1} + (-1)^m x'_{4m-1} & (i = m) \\ 0 & (i = 2m) \end{cases}$$

**Corollary 4.8.** In the notations of (3.6) and (2.6),  $\text{ch}: K^*(\text{Spin}(2n)) \rightarrow H^{**}(\text{Spin}(2n); \mathbb{Q})$  is given by

$$\begin{aligned} \text{ch}(\beta(u_k)) &= \sum_{i=1}^{n-1} \frac{(-1)^{i-1} 2^{q(n-1,i)}}{(2i-1)!} \phi(2n, k, 2i) \tilde{x}_{4i-1} + \\ &\quad \frac{(-1)^{n-1} (1+(-1)^n)}{(n-1)!} \phi(2n, k, n) \tilde{x}'_{2n-1}, \end{aligned}$$

$$\text{ch}(\beta(\Delta_{2n}^+)) =$$

$$\begin{aligned} &\sum_{i=1}^{n-1} \frac{(-1)^{i-1} 2^{q(n-1,i)}}{(2i-1)!} \left( \frac{1}{2^n} \sum_{\ell=0}^{[(n-2)/2]} \phi(2n, n-1-2\ell, 2i) \right) \tilde{x}_{4i-1} + \\ &\left( \frac{1}{2} + \frac{(-1)^{n-1} (1+(-1)^n)}{(n-1)!} \right) \left( \frac{1}{2^n} \sum_{\ell=0}^{[(n-2)/2]} \phi(2n, n-1-2\ell, n) \right) \tilde{x}'_{2n-1}, \end{aligned}$$

$$\text{ch}(\beta(\Delta_{2n}^-)) =$$

$$\begin{aligned} &\sum_{i=1}^{n-1} \frac{(-1)^{i-1} 2^{q(n-1,i)}}{(2i-1)!} \left( \frac{1}{2^n} \sum_{\ell=0}^{[(n-2)/2]} \phi(2n, n-1-2\ell, 2i) \right) \tilde{x}_{4i-1} + \\ &\left( \frac{-1}{2} + \frac{(-1)^{n-1} (1+(-1)^n)}{(n-1)!} \right) \left( \frac{1}{2^n} \sum_{\ell=0}^{[(n-2)/2]} \phi(2n, n-1-2\ell, n) \right) \tilde{x}'_{2n-1}. \end{aligned}$$

**Corollary 4.9.**  $\tilde{j}_{2n}^*: H^*(\text{Spin}(2n+1); \mathbb{Z}) \rightarrow H^*(\text{Spin}(2n); \mathbb{Z})$  satisfies:

(i) if  $n = 2m + 1$ ,

$$\tilde{j}_{4m+2}^*(\tilde{x}_{4i-1}) = \begin{cases} \tilde{x}_{4i-1} & (i = 1, \dots, 2m) \\ 0 & (i = 2m+1) \end{cases};$$

(ii) if  $n = 2m$ ,

$$\tilde{j}_{4m}^*(\tilde{x}_{4i-1}) = \begin{cases} \tilde{x}_{4i-1} & (i = 1, \dots, 2m-1 \text{ and } i \neq m) \\ \tilde{x}_{4m-1} + (-1)^m \tilde{x}'_{4m-1} & (i = m \text{ and } m \notin \mathcal{P}) \\ 2\tilde{x}_{4m-1} + (-1)^m \tilde{x}'_{4m-1} & (i = m \text{ and } m \in \mathcal{P}) \\ 0 & (i = 2m) \end{cases}$$

**Remark.** Let  $\phi_t: N \times N \rightarrow Z$  be a function defined by

$$\phi_t(n, q) = \sum_{k=1}^n \phi(n, k, q) \quad \text{for } n, q \in N.$$

Then  $\phi_t(n, 1) = 2^{n-1}$ ,  $\phi_t(n, 2i) = 2\phi_t(n-1, 2i)$  and  $\phi_t(n, 2i+1) = 0$  for  $i = 1, \dots, [(n-1)/2]$ . With this notation we find that

$$2 \sum_{k=1}^n \phi(2n+1, k, 2i) = \phi_t(2n+1, 2i) \quad \text{for } i = 1, \dots, n$$

and

$$\begin{aligned} 4 \sum_{\ell=0}^{[(n-2)/2]} \phi(2n, n-1-2\ell, 2i) &= \phi_t(2n, 2i) = \\ 2\phi(2n, n, 2i) + 4 \sum_{k=1}^{[(n-1)/2]} \phi(2n, n-2k, 2i) &\quad \text{for } i = 1, \dots, n-1 \end{aligned}$$

(cf. Theorems 4.3 and 4.5).

Finally, we display a list of  $\text{ch}: K^*(G) \rightarrow H^{**}(G; \mathbb{Q})$  for  $G = SO(n)$  and  $\text{Spin}(n)$  with  $3 \leq n \leq 9$ :

$SO(3)$

$$\text{ch}(\varepsilon_3) = x_3$$

$SO(4)$

$$\text{ch}(\delta_4) = x_3'$$

$$\text{ch}(\varepsilon_4) = x_3$$

$SO(5)$

$$\text{ch}(\mathcal{B}(u_1')) = 2x_3 - \frac{1}{3}x_7$$

$\text{Spin}(3)$

$$\text{ch}(\mathcal{B}(\Delta_3)) = \tilde{x}_3$$

$\text{Spin}(4)$

$$\text{ch}(\mathcal{B}(\Delta_4^+)) = \tilde{x}_3$$

$$\text{ch}(\mathcal{B}(\Delta_4^-)) = \tilde{x}_3 - \tilde{x}_3'$$

$\text{Spin}(5)$

$$\text{ch}(\mathcal{B}(u_1')) = 2\tilde{x}_3 - \frac{2}{3}\tilde{x}_7$$

$$\text{ch}(\varepsilon_5) = 2x_3 + \frac{1}{6}x_7$$

SO(6)

$$\text{ch}(\mathcal{B}(\mu_1)) = 2x_3 - \frac{1}{3}x_7$$

$$\text{ch}(\delta_6) = x'_5$$

$$\text{ch}(\varepsilon_6) = 2x_3 + x'_5 + \frac{1}{6}x_7$$

SO(7)

$$\text{ch}(\mathcal{B}(\mu'_1)) = 2x_3 - \frac{1}{3}x_7 + \frac{1}{60}x_{11}$$

$$\text{ch}(\mathcal{B}(\mu'_2)) = 10x_3 + \frac{1}{3}x_7 - \frac{5}{12}x_{11}$$

$$\text{ch}(\varepsilon_7) = 4x_3 + \frac{1}{3}x_7 + \frac{1}{30}x_{11}$$

Spin(7)

$$\text{ch}(\mathcal{B}(\mu'_1)) = 2\tilde{x}_3 - \frac{2}{3}\tilde{x}_7 + \frac{1}{60}\tilde{x}_{11}$$

$$\text{ch}(\mathcal{B}(\mu'_2)) = 10\tilde{x}_3 + \frac{2}{3}\tilde{x}_7 - \frac{5}{12}\tilde{x}_{11}$$

$$\text{ch}(\mathcal{B}(\Delta_7)) = 2\tilde{x}_3 + \frac{1}{3}\tilde{x}_7 + \frac{1}{60}\tilde{x}_{11}$$

SO(8)

$$\text{ch}(\mathcal{B}(\mu_1)) = 2x_3 - \frac{1}{3}x_7 - \frac{1}{3}x'_7 + \frac{1}{60}x_{11}$$

$$\text{ch}(\mathcal{B}(\mu_2)) = 12x_3 - \frac{2}{5}x_{11}$$

$$\text{ch}(\delta_8) = x'_7$$

$$\text{ch}(\varepsilon_8) = 4x_3 + \frac{1}{3}x_7 + \frac{4}{3}x'_7 + \frac{1}{30}x_{11}$$

Spin(8)

$$\text{ch}(\mathcal{B}(\mu_1)) = 2\tilde{x}_3 - \frac{2}{3}\tilde{x}_7 - \frac{1}{3}\tilde{x}'_7 + \frac{1}{60}\tilde{x}_{11}$$

$$\text{ch}(\mathcal{B}(\mu_2)) = 12\tilde{x}_3 - \frac{2}{5}\tilde{x}_{11}$$

$$\text{ch}(\mathcal{B}(\Delta_8^+)) = 2\tilde{x}_3 + \frac{1}{3}\tilde{x}_7 + \frac{2}{3}\tilde{x}'_7 + \frac{1}{60}\tilde{x}_{11}$$

$$\text{ch}(\mathcal{B}(\Delta_8^-)) = 2\tilde{x}_3 + \frac{1}{3}\tilde{x}_7 - \frac{1}{3}\tilde{x}'_7 + \frac{1}{60}\tilde{x}_{11}$$

SO(9)

$$\text{ch}(\mathcal{B}(\mu'_1)) = 2x_3 - \frac{1}{3}x_7 + \frac{1}{60}x_{11} - \frac{1}{2520}x_{15}$$

$$\text{ch}(\mathcal{B}(\mu'_2)) = 14x_3 - \frac{1}{3}x_7 - \frac{23}{60}x_{11} + \frac{119}{2520}x_{15}$$

$$\text{ch}(\mathcal{B}(\mu'_3)) = 42x_3 + 3x_7 - \frac{3}{20}x_{11} - \frac{1071}{2520}x_{15}$$

$$\text{ch}(\mathcal{B}(\Delta_5)) = \tilde{x}_3 + \frac{1}{6}\tilde{x}_7$$

Spin(6)

$$\text{ch}(\mathcal{B}(\mu_1)) = 2\tilde{x}_3 + \tilde{x}'_5 - \frac{2}{3}\tilde{x}_7$$

$$\text{ch}(\mathcal{B}(\Delta_6^+)) = \tilde{x}_3 + \frac{1}{2}\tilde{x}'_5 + \frac{1}{6}\tilde{x}_7$$

$$\text{ch}(\mathcal{B}(\Delta_6^-)) = \tilde{x}_3 - \frac{1}{2}\tilde{x}'_5 + \frac{1}{6}\tilde{x}_7$$

$$\text{ch}(\varepsilon_9) = 8x_3 + \frac{2}{3}x_7 + \frac{1}{15}x_{11} + \frac{17}{2520}x_{15}$$

### Spin(9)

$$\text{ch}(\mathcal{B}(\mu'_1)) = 2\tilde{x}_3 - \frac{1}{3}\tilde{x}_7 + \frac{1}{60}\tilde{x}_{11} - \frac{1}{1260}\tilde{x}_{15}$$

$$\text{ch}(\mathcal{B}(\mu'_2)) = 14\tilde{x}_3 - \frac{1}{3}\tilde{x}_7 - \frac{23}{60}\tilde{x}_{11} + \frac{119}{1260}\tilde{x}_{15}$$

$$\text{ch}(\mathcal{B}(\mu'_3)) = 42\tilde{x}_3 + 3\tilde{x}_7 - \frac{3}{20}\tilde{x}_{11} - \frac{1071}{1260}\tilde{x}_{15}$$

$$\text{ch}(\Delta_9) = 4\tilde{x}_3 + \frac{1}{3}\tilde{x}_7 + \frac{1}{30}\tilde{x}_{11} + \frac{17}{2520}\tilde{x}_{15}.$$

### References

- [1] R. P. Held and U. Suter: Die Bestimmung der unitären K-Theorie von  $SO(n)$  mit Hilfe der Atiyah-Hirzebruch-Spektralreihe. Math. Z. 122 (1971), 33-52.
- [2] L. Hodgkin: On the K-theory of Lie groups. Topology 6 (1967), 1-36.
- [3] D. Husemoller: Fibre Bundles, 2nd edition. Graduate Texts in Math. vol. 20 (Springer-Verlag, 1974).
- [4] H. Toda and M. Mimura: Topology of Lie Groups I. (Kinokuniya Book Store, 1978). (in Japanese)
- [5] T. Watanabe: Chern characters on compact Lie groups of low rank. Osaka J. Math. 22 (1985), 463-488.
- [6] \_\_\_\_\_: The Chern character on the odd spinor groups, Publ. RIMS Kyoto Univ. 22 (1986), 513-525.
- [7] \_\_\_\_\_: On the Chern character of symmetric spaces related to  $SU(n)$ . (submitted to Math. Proc. Cambridge Philos. Soc.)
- [8] I. Yokota: Groups and Representations. (Shōkabō, 1973). (in Japanese)